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グラフの星部分グラフ分解

Star decomposition indexes of graphs

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We deal with finite simple graphs, which have neither multiple edges nor loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The arboricity $a(G)$ of G is the minimum integer n for which $E(G)$ can be decomposed into n forests. A formula for the arboricity of a graph was obtained by Nash-William [5],[6]. The formula is the following:

$$a(G) = \max \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$$

where the maximum is taken over all subgraphs H of G, and $\lceil x \rceil$ denotes the least integer not less than x. If we impose some conditions to forests, then we obtain new invariants. A graph is called a linear forest if each component of it is a path, and linear arboricity $\Xi(G)$ of G is defined to be the minimum n for which $E(G)$ can be decomposed into n linear forests. Some results on linear arboricity can be found in [1],[4]. We call a graph H a star if H is isomorphic to the complete bipartite graph $K_{1,n}$ for some n (Fig. 1). We call a graph G

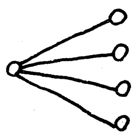


Figure 1. $K_{1,4}$.

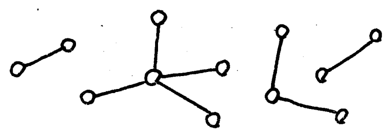


Figure 2. A star-forest.

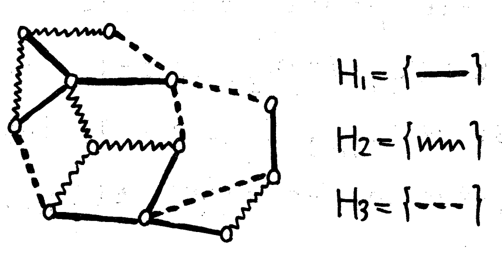


Figure 3. A graph G with $*(G)=3$.

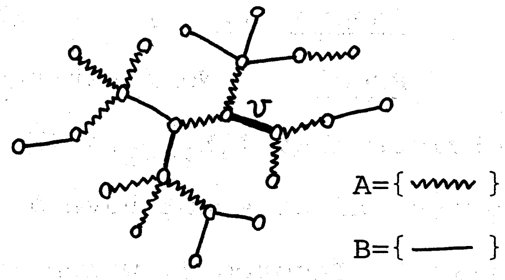


Figure 4. A and B.

a star-forest if each component of G is a star (Fig. 2). We define the star decomposition index $*(G)$ of G by the minimum n for which $E(G)$ can be decomposed into n star-forests (Fig. 3). In this paper we shall investigate star decomposition indexes.

We begin with the following easy result.

Proposition 1. Let T be a tree. If T is not a star, then $*(T)=2$.

Proof Let T be a tree that is not a star. Then it is obvious that $*(T)\geq 2$. Let $L(T)$ be the line graph of T (i.e. $V(L(T))=E(T)$ and two vertices of $L(T)$ are adjacent if and only if corresponding edges of T are adjacent.). For two vertices x and y of $L(T)$, we denote by $d(x,y)$ the distance between x and y in $L(T)$. Choose any vertex v of $L(T)$, and set

$$A=\{x\in V(L(T)) \mid d(v,x) \text{ is odd}\} \text{ and}$$

$$B=\{x\in V(L(T)) \mid d(v,x) \text{ id even}\}\ni v \text{ (Fig. 4).}$$

Then A and B are star-forests of T, and thus $*(T)\leq 2$. Therefore $*(T)=2$. □

By K_n and $K_{n,m}$, we denote the complete graph of order n and the complete graph of order n+m, respectively. Let A be a graph. Then an A-factor of a graph is its spanning subgraph each component of which is isomorphic to A.

<u>Theorem 1.</u> [2] $*(K_{2n-1})=*(K_{2n})=n+1$, where $n\geq 3$.

<u>Proof</u> We first show that $*(K_{2n})\geq *(K_{2n-1})\geq n+1$. It is obvious that $*(K_{2n-1})\leq *(K_{2n})$. Since K_{2n-1} is a $2(n-1)$-regular graph and does not have a $K_{1,n-1}$-factor, we obtain $*(K_{2n-1})\geq n+1$ by Theorem 3, which will be given later.

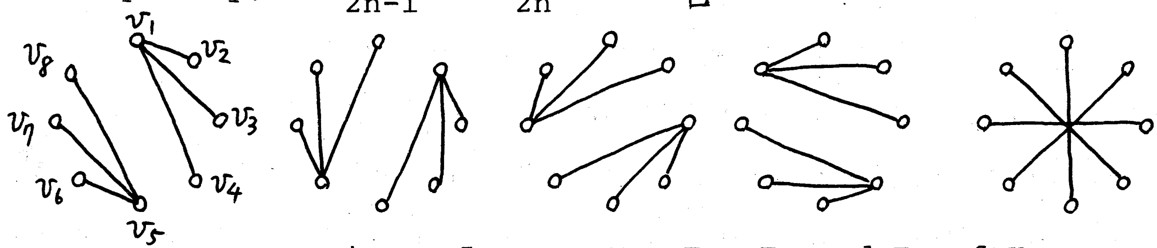
We next show that $*(K_{2n})\leq n+1$. Let $V(K_{2n})=\{v_1,v_2,\ldots,v_{2n}\}$ and put

$$F_t=\{v_tv_i \mid t<i'<t+n,\ i\equiv i' \pmod{2n}\}$$
$$\cup\{v_{n+t}v_j \mid n+t<j'<t+2n,\ j\equiv j' \pmod{2n}\}\subset E(K_{2n})$$

for $t=1,\ldots,n$, and define

$$F_{n+1}=\{v_1v_{n+1},v_2v_{n+2},\ldots,v_nv_{2n}\} \quad \text{(Fig. 5).}$$

Then $K_{2n}=F_1\cup F_2\cup \ldots \cup F_{n+1}$, and we conclude that $*(K_{2n})\leq n+1$. Consequently, $*(K_{2n-1})=*(K_{2n})=n+1$. □



<u>Figure 5.</u> F_1, F_2, F_3, F_4 and F_5 of K_8.

The star decomposition index of the complete bipartite graph $K_{n,n}$ was determined by Egawa, Fukuda, Nagoya and Urabe [3], and $*(K_{n,m})$ for some classes of n,m are obtained by Enomoto and etc.

<u>Theorem 2.</u> [3] $*(K_{2n,2n})=*(K_{2n-1,2n-1})=n+2$, where $n\geq 4$.

<u>Proof</u> We prove only that $*(K_{2n-1,2n-1})\leq *(K_{2n,2n})\leq n+2$. For the proof of $*(K_{2n-1,2n-1})\geq n+2$, the reader should refer to [3]. It is trivial that $*(K_{2n-1,2n-1})\leq *(K_{2n,2n})$. Let $V(K_{2n,2n})=\{a_1,\ldots,a_n,b_1,\ldots,b_n\}\cup\{c_1,\ldots,c_n,d_1,\ldots,d_n\}$. For every k, $1\leq k\leq n$, we define

$$F_k = \{a_k c_i,\ b_k d_i,\ a_i d_k,\ b_i c_k \mid 1 \le i \le n,\ i \ne k\},$$

and put

$$F_{n+1} = \{a_i c_i,\ a_i d_i \mid 1 \le i \le n\}, \text{ and}$$

$$F_{n+2} = \{b_i c_i,\ b_i d_i \mid 1 \le i \le n\} \quad \text{(Fig. 6)}.$$

Then $K_{2n,2n} = F_1 \cup F_2 \cup \dots \cup F_{n+2}$. Consequently, $*(K_{2n,2n}) \le n+2$.

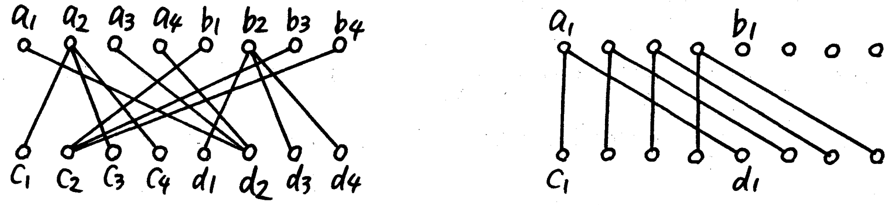


Figure 6. F_2 amd F_5 of $K_{8,8}$.

We write $d_G(v)$ for the degree of vertex v in G. A graph G is called an r-regular graph if $d_G(x)=r$ for all vertices x.

Theorem 3. Let G be a 2r-regular graph. Then

$$*(G) \ge r+1$$

with the equality if and only if G can be decomposed into r+1 edge-disjoint $K_{1,r}$-factors.

Proof Since $a(G) \ge |E(G)|/(|V(G)|-1) > r$, we have $*(G) \ge a(G) \ge r+1$. Suppose $*(G)=r+1$. Then G can be decomposed into r+1 star-forests $H_1, H_2, \dots, H_{r+1}$. We denote by $n_i(t)$ the number of components $K_{1,t}$ in H_i. Put $p=|V(G)|$ and $x_i=|V(G)| - |V(H)|$. Then we have

$$p = x_k + \sum_{j=1}^{2r} (j+1) n_k(j) \qquad \text{for all } k,\ 1 \le k \le r+1 \tag{1}$$

$$|E(H_k)| = \sum_{j\ge 1} j\, n_k(j) = p - (x_k + \sum_{j \ge 1} n_k(j)), \text{ and}$$

$$\sum_{x \in V(G)} d_G(x) = 2pr = 2|E(G)| = 2 \sum_{k=1}^{r+1} |E(H_k)|$$

$$= 2p(r+1) - 2 \sum_{k=1}^{r+1} \{x_k + \sum_{j\ge 1} n_k(j)\}$$

Therefore

$$p = \sum_{k=1}^{r+1} \{ x_k + \sum_{j\geq 1} n_k(j) \}. \tag{2}$$

The vertex of $K_{1,t}$ $(t\geq 2)$ with degree t is called the center of $K_{1,t}$. It follows that every vertex v of $K_{1,j}$ $(j<r)$ in H_1 must be the center of a component $K_{1,t}$ $(t\geq 2)$ in some H_k $(k\geq 2)$, since otherwise $2r=d_G(v)=d_{H_1}(v)+\sum_{k\geq 2} d_{H_k}(v) < r+r$, a contradiction.

Similarly, every end vertex of $K_{1,j}$ $(j\geq r)$ in H_1 is contained in the center of a component in some H_k $(k\geq 2)$. Hence

$$x_1 + \sum_{j=1}^{r-1} (j+1)n_1(j) + \sum_{j=r}^{2r} jn_1(j) \leq \sum_{k=2}^{r+1} (\sum_{t\geq 2} n_k(t)) \tag{3}$$

By substituting (3) into (2), we obtain

$$\begin{aligned} p &= x_1 + \sum_{j\geq 1} n_1(j) + \sum_{k=2}^{r+1} x_k + \sum_{k=2}^{r+1} (\sum_{j\geq 2} n_k(j)) + \sum_{k=2}^{r+1} n_k(1) \\ &\geq x_1 + \sum_{j\geq 1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{j=1}^{r-1} (j+1)n_1(j) + \sum_{j=r}^{2r} jn_1(j) \\ &\qquad + \sum_{k=2}^{r+1} n_k(1) \\ &= x_1 + \sum_{j\geq 1} (j+1)n_1(j) + \sum_{j=1}^{r-1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{k=2}^{r+1} n_k(1) \\ &= p + \sum_{j=1}^{r-1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{k=2}^{r+1} n_k(1). \qquad \text{(by (1))} \end{aligned}$$

Hence $n_1(j)=0$ for every j, $1\leq j\leq r-1$, and $x_1=\ldots=x_{r+1}=0$. We can similarly show that $n_k(j)=0$ for all k, j $(k\geq 2$ and $j\leq r-1)$. Therefore, each component of H_k is $K_{1,t}$ $(t\geq r)$, and H_k is a spanning subgraph of G. If $d_{H_k}(v)\geq r+1$ for some $k\geq 1$ and $v\in V(G)$, then $2r=d_G(v) = \sum_{t\geq 1} d_{H_t}(v)\geq r+1+r=2r+1$, a contradiction. Consequently, each H_k has no $K_{1,t}$ for $t\geq r+1$, and we conclude that every H_k is a $K_{1,r}$-factor of G. Hence the proof is complete. □

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The next theorem can be proved by the same argument in the proof of Theorem 3.

Theorem 4. Let G be a (2r+1)-regular graph. Then $*(G) \geq r+2$.

By Theorems 3 and 4, we have

r	2	3	4	5	6
*(G) of r-regular graph G	2	3	3,4	4,5	4,5,6

Note that the existence of a 5-regular graph G_1 with $*(G_1)=5$ and of a 6-regular graph G_2 with $*(G_2)=6$ is unknown.

A triagle cluster is a connected graph whose edges partition into disjoint triangles with the property that any two triangles have at most one vertex in common and if such a vertex exists, then it is a cut vertex of the cluster (Fig. 7).

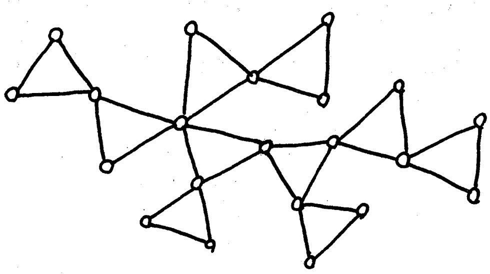

Figure 7. A triangle cluster.

Proposition 2. (Fukuda [5]) Let TC be a triangle cluster. Then

$$*(TC) = \begin{cases} 2 & \text{if every triangle has a vertex of degree 2} \\ 3 & \text{otherwise.} \end{cases}$$

References

[1] J. Akiyama, Factorization and linear arboricity of graphs, Doctor thesis, Science University of Tokyo, (1982).

[2] J. Akiyama and M. Kano, Path factors of graphs, Graphs and Applications (Proc. of the first Colorado symposium of graph theory, Wiley) to appear.

[3] Y. Egawa, T. Fukuda, S. Nagoya and M. Urabe, A decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components, to appear.

[4] H. Enomoto, The linear arboricity of 5-regular graphs, to appear.

[5] T. Fukuda, 修士論文，電通大情報数理工学科 (1984)、

[6] C. Nash-William, Decomposition of finite graphs into forests J. London Math. Soc. 39 (1964) 12.

[7] C. Nash-William, Edge-disjoint spanning trees of finite graphs, J. London Math. Sco. 36 (1961) 445-450.